Differential Geometry IV: General Relativity

6.1 Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold. We will define the *Riemann curvature tensor* $R : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ by

$$R(X, Y)Z \doteq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

(a) Show that R is indeed a $(1, 3)$-tensor field which is antisymmetric in its first two arguments.

Let V be a Killing vector field on $(\mathcal{M}, g)$ and $\gamma : (a, b) \to \mathcal{M}$ a geodesic for g.

(b) Prove that, for any vector field Z along γ:

$$g\big(\nabla_{\dot\gamma} \nabla_{\dot\gamma} V,\, Z\big) + g\big(R(\dot\gamma, Z)V, \dot\gamma)\big) = 0.$$

Deduce that the components of V^k of V in any local coordinate system around a given point on γ satisfy a second order linear ODE along γ.

(c) Show that if $V|_p = 0$ and $\nabla V|_p = 0$ for some $p \in \mathcal{M}$, then $V = 0$ on the whole connected component of $\mathcal{M}$ containing p.

(d) What is the maximum dimension of the Lie algebra of Killing fields on a connected Lorentzian manifold of dimension $n + 1$? Compare it with the dimension of the Killing algebra on Minkowski spacetime.

6.2 (a) Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let V be a Killing vector field on $(\mathcal{M}, g)$. Show that for any geodesic $\gamma : I \to \mathcal{M}$, the inner product $g(\dot\gamma, V)$ is constant along V.

*(b) Let $\mathcal{M} = \mathbb{R} \times \bar{\mathcal{M}}$ be a product manifold equipped with a Lorentzian metric g of the form

$$g = -f \cdot dt \otimes dt + dt \otimes \omega + \omega \otimes dt + \bar g,$$

where

- $t : \mathcal{M} = \mathbb{R} \times \bar{\mathcal{M}} \to \mathbb{R}$ is the projection on the first factor.
- f, h are smooth functions on $\bar{\mathcal{M}}$.
- ω is an 1-form on $\bar{\mathcal{M}}$.
- $\bar g$ is a Riemannian metric on $\bar{\mathcal{M}}$.

Let $\mathfrak{E} \subset \bar{\mathcal{M}}$ be the set where $f < 0$. Show that, for every $p \in \mathbb{R} \times \mathfrak{E}$, there exists a maximally extended **null** geodesic $\gamma : (a, b) \to \mathcal{M}$ for g with $\gamma(0) = p$ which does not escape $\mathbb{R} \times \mathfrak{E}$ (i.e. $\gamma(s) \in \mathbb{R} \times \mathfrak{E}$ for all $s \in (a, b)$).

6.3 Consider the manifold $\mathcal{M} = \mathbb{R} \times \mathbb{R}$ equipped with the Lorentzian metric

$$g_{AdS} = -(1 + r^2)dt^2 + \frac{1}{1 + r^2} dr^2$$

(this is known as the $1+1$ dimensional *Anti-de Sitter metric*). Note that $(\mathcal{M}, g)$ can be thought of as the universal Lorentzian cover of the (topological) cylinder $\mathcal{S} = \{-x^2 - y^2 + r^2 = +1\}$ in the pseudo-Euclidean space $(\mathbb{R}^{2+1}, \eta_{(2,1)})$, where $\eta_{(2,1)} = -dx^2 - dy^2 + dr^2$.

(a) Show that $(\mathcal{M}, g_{AdS})$ is timelike geodesically complete, i.e. that all timelike geodesics of g_{AdS} can be extended on the whole of $\mathbb{R}$.

(b) Show that *all* timelike geodesics γ passing through $(t, r) = (0, 0)$ also pass through $(t, r) = (k\pi, 0)$, $k \in \mathbb{Z}$. Show that there exists a point $p \in \mathcal{I}^+[(0, 0)]$ such that $(0, 0)$ and p cannot be connected with a timelike geodesic.

6.4 Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let $S \subset \mathcal{M}$ be a smooth *null* hypersurface. Let L be a non-zero vector field along S such that, for every $p \in S$, $L|_p \perp T_pS$.

(a) Show that L is tangent to S (i.e. $L|_p \in T_pS$ for all $p \in S$).

(b) Show that, for any two vector fields X, Y tangent to S, we have

$$g\big(\nabla_X L, Y\big) = g\big(\nabla_Y L, X\big).$$

(*Hint: You might want to use the fact that, if V, W are two vector fields tangent to a submanifold $\mathcal{N} \subset \mathcal{M}$, then $[V, W]$ is also tangent to $\mathcal{N}$.*)

(c) Using the above formula, show that $\nabla_L L|_p \perp T_pS$ for all $p \in S$. Show that

$$\nabla_L L = \kappa L \quad \text{for some } \kappa : S \to \mathbb{R}.$$

Deduce that the integral curves of L in S are (up to reparametrization) null geodesics. Infer that any null hypersurface of $(\mathbb{R}^{1+n}, \eta)$ is a union of null lines.

(d) Assume, in addition, that there exists a Killing vector field V on $(\mathcal{M}, g)$ such that V is collinear with L along S (such a null hypersurface is called a *Killing horizon*). By replacing L with V in the above arguments, we infer that

$$\nabla_V V|_S = \kappa V|_S.$$

Show that the function κ above is a *constant* along the null generators of S, i.e. $V(\kappa) = 0$. This is the *surface gravity* of the Killing horizon.

(e) Show that the hyperplane $H = \{x = t\}$ in $(\mathbb{R}^{3+1}, \eta)$ (with the usual Cartesian coordinate system (t, x, y, z)) is a Killing horizon (and find the corresponding Killing vector field of $(\mathbb{R}^{3+1}, \eta)$. Can you compute its surface gravity?